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Simple Sufficient Conditions for the Existence of a Utility Function Representing Consumer Preferences

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Abstract: *We provide simple sufficient conditions for the numerical representation of consumer preferences used in demand theory.*

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1. Introduction

This note based on Appendix A of Demand, Demand Curves and Consumer Surplus (Lahiri (2022)) is a simple alternative to the usual result on numerical representation of consumer preferences by Wold (1943) that is used in demand theory. The usual result along with a proof is available as a proposition on page 82 of Varian (1978).

The appendix A that we mention above (if referred to at all) is meant to precede the matter in Chapter 4 (Classical Demand Theory) in Lahiri (2022).

The contribution of the paper is largely pedagogical. Instead of assuming consumer preferences are weakly monotonic and continuous- the latter being a continuity concept different from continuity of functions- so that continuous and weakly monotonic numerical representations exist, we assume that preferences are weakly monotonic and satisfy a property called straight-line property. Then preferences have weakly monotonic numerical representations. If and when we require continuity of the utility function, we introduce this as an additional assumption, but this being a continuity assumption about functions, it is much easier for students to conceptualize than conceptualizing continuous preferences. Further, as a result of our assumption on preferences our representation theorem does not depend on any topological requirements and is applicable in more general contexts than where several existing representation theorems, including that of Wold (1943), apply or are potentially applicable to.

2. The Model and the Sufficient Conditions

As in chapter 1 of Lahiri (2022), consider an economy in which there is a commodity called **money** and L non-monetary commodities (goods), where L is a positive integer.

The **commodity space** is $\mathbb{R}^{L+1}$ any element of which is called a **commodity bundle.** A typical commodity bundle will be, denoted as $x \in \mathbb{R}^{L+1}$, where x_k – the k^{th} coordinate of x- denotes quantities of the k^{th} good for $k \in \{1,\ldots,L\}$, and x_{L+1} denotes quantity of money.

A consumer is an economic agent who derives satisfaction or dis-satisfaction from consuming commodity bundles.

The **consumption set** of the consumer is a non-empty subset X of $\mathbb{R}^{L+1}$ and an element of the consumption set is said to be a **consumption bundle**.

Often, but not invariably it is assumed that the consumption set is $\mathbb{R}^{L+1}_{+}$.

We assume that the consumer has preferences over consumption bundles given by a **relation** on X denoted by $\succcurlyeq$ such that for $x,y \in X$: $x \succcurlyeq y$ <u>means</u> the consumption bundle x is **at least as good as** the consumption bundle y. The asymmetric part of $\succcurlyeq$ is denoted by $\succ$ (i.e. for $x,y \in X$: $x \succ y$ if and only if [$x \succcurlyeq y$ and it is not the case that $y \succcurlyeq x$]) and is said to be the **strict preference relation** of the consumer, so that for $x,y \in X$: $x \succ y$ <u>means</u> the consumption bundle x is **strictly preferred to** the consumption bundle y. The symmetric part of $\succcurlyeq$ is denoted by $\sim$ (i.e. for $x,y \in X$: $x \sim y$ if and only if [$x \succcurlyeq y$ and $y \succcurlyeq x$]) and is said to be the **no difference relation** of the consumer so that for $x,y \in X$: $x \sim y$ <u>means</u> the consumption bundle x is **no different from** the consumption bundle y.

The relation $\succcurlyeq$ on X is said to be:

(i) **reflexive** if for all $x \in X$ it is the case that $x \succcurlyeq x$;

(ii) **transitive** if for all $x, y, z \in X$: [$x \succcurlyeq y$ and $y \succcurlyeq z$] implies [$x \succcurlyeq z$];

(iii) **complete/total/connected** if for all $x, y \in X$ with $x \neq y$: <u>either</u> $x \succcurlyeq y$ <u>or</u> $y \succcurlyeq x$];

(iv) **a preference relation** if it is reflexive, transitive and connected.

A preference relation $\succcurlyeq$ on X is said to be **weakly monotonic** if for all $x,y \in X$: [$x >> y$, i.e. $x-y \in \mathbb{R}^{L+1}_{++}$] implies [$x \succ y$];

A preference relation $\succcurlyeq$ on X is said to satisfy **straight-line property** if there exists $\Omega \in \mathbb{R}^{L+1}_{++}$ (possibly depending on X and $\succcurlyeq$) such that for all $x \in X$ there exists an $\alpha \in \mathbb{R}$ for which (a) $\alpha\Omega \in X$ and (b) $x \sim \alpha\Omega$.

Note: We call the above property "straight-line property" because given any $\Omega \in \mathbb{R}^{L+1}_{++}$, the set $\{\alpha\Omega \mid \alpha \in \mathbb{R}\}$ is a straight line through the origin.

A preference relation $\succcurlyeq$ on X is said to be **numerically representable** if there exists a function $u: X \to \mathbb{R}$ such that for all $x, y \in X$: $[x \succcurlyeq y]$ if and only if $[u(x) \geq u(y)]$.

Note: If $X = \mathbb{R}^{L+1}_{+}$, then for all $\alpha \in \mathbb{R}$ and $\Omega \in \mathbb{R}^{L+1}_{++}$: $[\alpha\Omega \in X]$ if and only if $[\alpha \in \mathbb{R}_{+}]$.

If $\succcurlyeq$ is numerically representable and if $u: X \to \mathbb{R}$ is such that for all $x, y \in X$: $[x \succcurlyeq y]$ if and only if $[u(x) \geq u(y)]$, then u is said to be a **utility function for** (or **numerical representation of**) $\succcurlyeq$ (on X).

Further if f: range(u) $\to \mathbb{R}$ is a strictly increasing function (i.e. for all α, $\beta \in$ range(u), $\alpha > \beta$ implies $f(\alpha) > f(\beta)$) and $v: X \to \mathbb{R}$ satisfies $v(x) = f(u(x))$ for all $x \in X$, then v is also a utility function for $\succcurlyeq$.

3. The Main Result

We provide below our main result.

Theorem: If a preference relation $\succcurlyeq$ is weakly monotonic and satisfies straight-line property then it is numerically representable.

Proof: Suppose $\succcurlyeq$ is weakly monotonic and satisfies straight-line property. Then $\Omega \in \mathbb{R}^{L+1}_{++}$ there exists $\Omega \in \mathbb{R}^{L+1}_{++}$ such that for all $x \in \mathbb{R}$ there exists an α satisfying $\alpha\Omega \in X$ and $x \sim \alpha\Omega$ and by weak monotonicity, such an α must be unique. Let u(x) be the unique real number such that $x \sim u(x)\Omega$.

Let $x, y \in X$.

Suppose $x \succcurlyeq y$. If $u(y) > u(x)$ then by weak monotonicity, $u(y)\Omega \succ u(x)\Omega$. Since, $x \sim u(x)\Omega$ and $y \sim u(y)\Omega$, then by transitivity of $\succcurlyeq$ we get $y \succ x$, leading to a contradiction. Thus, $u(x) \geq u(y)$.

Now suppose $u(x) \geq u(y)$.

If $u(x) = u(y)$, then $x \sim u(x)\Omega$, $y \sim u(y)e$ and by reflexivity $u(x)\Omega \sim u(y)\Omega$. Thus, by transitivity of $\succcurlyeq$ we get $x \sim y$, which implies $x \succcurlyeq y$.

If $u(x) > u(y)$, then $x \sim u(x)\Omega$, $y \sim u(y)\Omega$ and by weak monotonicity $u(x)\Omega \succ u(y)\Omega$. Thus, by transitivity of $\succcurlyeq$ we get $x \succ y$, which implies $x \succcurlyeq y$. Q.E.D.

Note: It is easy to see that, lexicographic preferences on $\mathbb{R}^{L+1}_{+}$(page 83, example 3.2 of Varian (1978)) do not satisfy straight-line property, since for all $x,y \in \mathbb{R}_+^{L+1}$ with $x \neq y$, either x is strictly preferred to y or y is strictly preferred to x. It is well known that lexicographic preferences are not representable by a utility function.

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References

Lahiri, S. (2022). "Demand, Demand Curves and Consumer Surplus", *https://drive.google.com/file/d/1Ze-1s_JGisMIFXuGeU0LdLefYzJvQH5o/view*

Varian, H. (1978). "Microeconomic Analysis", W.W. Norton and Company, INC.

Wold, H. (1943). "A Synthesis of Pure Demand Analysis', I-III. Skandinavisk Aktuarietidskrift 26, 27.